

# INTEGRAL REPRESENTATIONS OF SOLUTIONS OF PERIODIC ELLIPTIC EQUATIONS

PETER KUCHMENT

*Dedicated to Stas Molchanov on the occasion of his 65th birthday*

**ABSTRACT.** The paper discusses relations between the structure of the complex Fermi surface below the spectrum of a second order periodic elliptic equation and integral representations of certain classes of its solutions. These integral representations are analogs of those previously obtained by S. Agmon, S. Helgason, and other authors for solutions of the Helmholtz equation (i.e., for generalized eigenfunctions of Laplace operator). In a previous joint work with Y. Pinchover we described all solutions that can be represented as integrals of positive Bloch solutions over the imaginary Fermi surface, with a hyperfunction as a “measure”. Here we characterize the class of solutions such that the corresponding hyperfunction is a distribution on the Fermi surface.

## 1. INTRODUCTION

This paper is devoted to integral representations of solutions of second order elliptic periodic differential equations. These representations are analogs of those for solutions of the Helmholtz equation in  $\mathbb{R}^n$

$$(1.1) \quad -\Delta u - k^2 u = 0 \quad \text{in } \mathbb{R}^n,$$

where  $k \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ . Such representations have been obtained by S. Helgason [15, 16], M. Hashizume et al. [14], M. Morimoto [30], and S. Agmon [2, 3]. In these results, solutions were expanded into exponential ones

$$(1.2) \quad e_\xi(x) := \exp(i\xi \cdot x).$$

Here

$$(1.3) \quad \xi \in S := \{\xi \in \mathbb{C}^n \mid \xi = k\omega, \omega \in S^{n-1} \subset \mathbb{R}^n\},$$

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and  $\xi \cdot x = \sum_{j=1}^n \xi_j x_j$ . These expansions can be written as

$$(1.4) \quad u(x) = \int_S e_\xi(x) d\phi(\xi) = \langle \phi(\xi), e_\xi(x) \rangle,$$

where  $\phi(\xi)$  is a functional on the sphere  $S$ . In particular, it was understood what classes of solutions correspond to different classes of functionals (e.g., hyperfunctions, distributions, measures) [2, 3].

Such representations are related to the L. Ehrenpreis' *fundamental principle* [11, 32] for constant coefficient operators, which in the particular case of (1.1) claims that any solution of (1.1) can be represented as an integral with respect to the parameter  $\xi$  of the exponential solutions

$$(1.5) \quad e_\xi(x) := \exp(i\xi \cdot x) \quad \xi \in \Sigma.$$

Here

$$\Sigma := \{ \xi \in \mathbb{C}^n \mid \xi^2 = k^2 \},$$

is the characteristic variety of the operator in the left hand side of (1.1) (see the details and more precise formulation in [11, 32]). The representation (1.5) is highly non-unique, due to existence of functionals orthogonal to analytic functions on  $\Sigma$ . On the other hand,  $\Sigma$  is an analytic subset of  $\mathbb{C}^n$ , uniquely determined for  $k \neq 0$  by its spherical subset  $S$ . Thus, one can expect the possibility of a unique representation like the one in (1.4). It is crucial here that  $\Sigma$  is irreducible and that  $S$  is sufficiently massive, so  $S$  determines  $\Sigma$  uniquely (otherwise it would not be possible to obtain the representation of all solutions using only  $\xi \in S$ ). Moreover,  $S$  is a rather simple analytic manifold. This enables one to obtain rather explicit descriptions of the needed spaces of test functions and functionals.

In this paper, we consider the case of second order periodic elliptic equations (see the exact description of the class of equations in the next section). For such (and more general) periodic equations, an analog of the “fundamental principle” was obtained in [19, 33] for solutions with some growth restrictions. Here, instead of exponential solutions one needs to use the so called *Floquet-Bloch* solutions. The analog of the characteristic manifold  $\Sigma$  is the Fermi surface  $F$  (see [4, 19, 35] and definitions 1 and 2 below for these notions). However, finding an analog of (1.4) for the periodic case is far from being straightforward. In particular, one wonders what should be the analog of the sphere  $S$ . It is natural to expect that when zero belongs to the spectrum of the operator, one might try to use the *real part* of the Fermi surface, while if zero does not belong to the spectrum, the *imaginary part* might be appropriate. Due to the complicated structure of the Fermi

surface, this idea is not easy to implement. As it was shown in the joint work [21] with Y. Pinchover, for second order elliptic equations with *positive* generalized principal eigenvalue  $\Lambda_0$  (see (2.9)), an appropriate variety is provided by the analysis of the cone of positive solutions done by S. Agmon and by V. Lin and Y. Pinchover [1, 26, 19]. No results of this type are known so far above  $\Lambda_0$ . Another difficulty is in proving irreducibility of the Fermi surface  $F$ , which happens to be a very hard problem (it also arises in many other spectral considerations [6, 13, 18, 20, 23, 24, 25]). Fortunately, as the reader can see from this paper and from [21], by appropriately restricting the growth of the solutions, we manage to work near a single irreducible component of  $F$  and hence avoid proving the irreducibility of  $F$ . Consequently, we prove a representation theorem (Theorem 13) that characterizes classes of the solutions that have integral expansion analogous to (1.4) into *positive* Bloch solutions with hyperfunctions or distributions as “measures”. The hyperfunction case was investigated in [21] and is presented here without a proof. The distribution result that we could not obtain in [21] is new and is proved in the present paper. In order to prove it, additional analytic techniques need to be involved, in particular results on peak sets in  $A^\infty$  functional algebras in several complex variables [8, 9]. We are grateful to A. Tumanov for pointing us to the relevant literature.

The proofs of the results of this paper are based upon the techniques of the Floquet theory developed in [19] (the reader can find all necessary preliminary information in the next section). The methods that were used to prove the “fundamental principle” [11, 33] provide a crucial technical approach. In particular, solutions of the equation are treated in the dual sense, i.e., as functionals on appropriate function spaces that are orthogonal to the range of the dual operator.

The outline of the paper is as follows. The next section introduces necessary notations and preliminary results from the Floquet theory and the theory of positive solutions of periodic elliptic equations. It mostly (but not entirely) repeats the corresponding sections from [21, 22] and is included for the reader’s convenience. Section 3 contains the proof of the integral representation (Theorem 13) that describes the sets of solutions allowing integral representations with distributional and hyperfunction “measures”. The last Section contains acknowledgments.

## 2. NOTATIONS AND PRELIMINARY RESULTS

Due to the nature of this section, most of it repeats some parts of [19, 21, 22]. We regret the necessity of doing this, but otherwise reading the rest of the paper would probably become impossible without constant referring to [21].

In this paper we consider second order elliptic operators on  $\mathbb{R}^n$  with *real* periodic coefficients of the form

$$(2.1) \quad P(x, \partial) = - \sum_{i,j=1}^n a_{ij}(x) \partial_i \partial_j + \sum_{i=1}^n b_i(x) \partial_i + c(x).$$

It is assumed that the uniform ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x) \zeta_i \zeta_j \geq a \sum_{i=1}^n \zeta_i^2$$

is satisfied for all  $x, \zeta \in \mathbb{R}^n$ , where  $a$  is a positive constant. In the notation  $P(x, D)$  we used the standard convention  $D = -i\partial/\partial x$ .

We will assume sufficient smoothness of the coefficients, namely that  $a_{ij} \in C^2(\mathbb{R}^n)$ ,  $b_i \in C^1(\mathbb{R}^n)$  and  $c \in C(\mathbb{R}^n)$ . In fact, it is sufficient to assume that both the operator  $P$  and its dual (the formal adjoint)  $P^*$  have Hölder continuous coefficients<sup>1</sup>. Here the duality is provided by the bilinear (rather than the sesquilinear) form

$$\langle g, f \rangle := \int_{\mathbb{R}^n} f(x) g(x) dx.$$

So, the dual operator  $P^*$  has similar properties to the ones of  $P$ .

The coefficients of  $P$  are assumed to be periodic with respect to a lattice  $\Gamma$  in  $\mathbb{R}^n$ . In what follows, the particular choice of the lattice is irrelevant and can always be reduced by change of variables to the case  $\Gamma = \mathbb{Z}^n$ , which we will assume from now on. We will always use the word “periodic” in the meaning of “ $\Gamma$ -periodic”.

We now introduce some standard notions and results from Floquet theory of periodic PDEs [4, 10, 19, 21, 22, 35].

We denote by  $K = [0, 1]^n$  the standard fundamental domain (the *Wigner-Seitz cell*) of the lattice  $\Gamma = \mathbb{Z}^n$ , and by  $B = [-\pi, \pi]^n$  the *first Brillouin zone*, which is a fundamental domain of the reciprocal (dual) lattice  $\Gamma^* = (2\pi\mathbb{Z})^n$ . We naturally identify  $\Gamma$ -periodic functions with functions on  $\mathbb{T}^n = \mathbb{R}^n/\Gamma$ .

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<sup>1</sup>See [21, Section 6]. We only need that both operators  $P$  and  $P^*$  define Fredholm mappings between the Sobolev space  $H^2(\mathbb{T}^n)$  and  $L_2(\mathbb{T}^n)$ , where  $\mathbb{T}^n = \mathbb{R}^n/\Gamma$ .

**Definition 1.** A function  $u(x)$  representable as a finite sum of the form

$$(2.2) \quad u(x) = e^{ik \cdot x} \left( \sum_{j=(j_1, \dots, j_n) \in \mathbb{Z}_+^n} x^j p_j(x) \right)$$

with nonzero  $\Gamma$ -periodic functions  $p_j(x)$  is called a *Floquet function with a quasimomentum*  $k \in \mathbb{C}^n$ . Here  $x^j = x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$ . The maximum value of  $|j| = \sum_{l=1}^n j_l$  in the representation (2.2) is said to be the *order* of the Floquet function. Floquet functions of zero order are called *Bloch functions*.

The set introduced below plays in the periodic case the role of the characteristic variety  $\Sigma$  for constant coefficient operators.

**Definition 2.** The (complex) *Fermi surface*  $F_P$  of the operator  $P$  (at the zero energy level) consists of all vectors  $k \in \mathbb{C}^n$  (called *quasimomenta*) such that the equation  $Pu = 0$  has a nonzero Bloch solution  $u(x) = e^{ik \cdot x} p(x)$ , where  $p(x)$  is a  $\Gamma$ -periodic function.

Introducing a spectral parameter  $\lambda$ , one arrives at the notion of the *Bloch variety*:

**Definition 3.** The (complex) *Bloch variety*  $B_P$  of the operator  $P$  consists of all pairs  $(k, \lambda) \in \mathbb{C}^{n+1}$  such that the equation  $Pu = \lambda u$  has a nonzero Bloch solution  $u(x) = e^{ik \cdot x} p(x)$  with the quasimomentum  $k$ .

The Bloch variety  $B_P$  can be treated as the graph of a multivalued function  $\lambda(k)$  (so called *dispersion relation*) that assigns to any quasimomentum  $k$  the spectrum of the operator  $P(x, D + k)$  on the torus  $\mathbb{T}^n$ . Since for operators of the type (2.1), these spectra are known to be discrete (as in particular the discussion below will show), we can single out continuous branches  $\lambda_j$  of  $\lambda(k)$ . These branches are usually called the *band functions* (see [4, 35, 19]). The Fermi surfaces now become the level sets of the dispersion relation.

**Lemma 4** ([19, Theorems 3.1.5, 3.1.7 and 4.4.2]). (1) *The Fermi and Bloch varieties are the sets of all zeros of entire functions of a finite order in  $\mathbb{C}^n$  and  $\mathbb{C}^{n+1}$ , respectively.*

(2) *A quasimomentum  $k$  belongs to  $F_{P^*}$  if and only if  $-k \in F_P$ . Analogously,  $(k, \lambda) \in B_{P^*}$  if and only if  $(-k, \lambda) \in B_P$ . In other words, the dispersion relations  $\lambda(k)$  and  $\lambda^*(k)$  for the operators  $P$  and  $P^*$  are related as follows:*

$$(2.3) \quad \lambda^*(k) = \lambda(-k).$$

The Fermi surface  $F_P$  is periodic with respect to the reciprocal lattice  $\Gamma^* = (2\pi\mathbb{Z})^n$ . It is often convenient to factor out this periodicity by considering the (analytic) exponential mapping  $\rho : \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n$ , where

$$z = \rho(k) = \rho(k_1, \dots, k_n) = (\exp ik_1, \dots, \exp ik_n).$$

This mapping can be identified with the quotient map  $\mathbb{C}^n \rightarrow \mathbb{C}^n/\Gamma^*$ . We also introduce the complex torus

$$(2.4) \quad T := \rho(\mathbb{R}^n) = \{z \in \mathbb{C}^n \mid |z_j| = 1, j = 1, 2, \dots, n\}.$$

**Definition 5.** We call the image  $\Phi_P := \rho(F_P)$  under the mapping  $\rho$  of the Fermi surface  $F_P$  the *Floquet surface* of the operator  $P$ .

In the Floquet theory for PDEs, this Floquet surface is the set of all Floquet multipliers of Floquet-Bloch solutions of the equation  $Pu = 0$ .

The following analog  $\mathcal{U}$  of the Fourier transform (see [19, Section 2.2], [35]), which we will call the *Floquet transform*<sup>2</sup>, is the main tool in the Floquet theory:

$$(2.5) \quad f(x) \rightarrow \mathcal{U}f(z, x) := \sum_{\gamma \in \Gamma} f(x - \gamma) z^\gamma \quad z \in (\mathbb{C}^*)^n.$$

It is often convenient to use for the Floquet transform  $\mathcal{U}$  the quasi-momentum coordinate  $k$  instead of the multiplier  $z = \rho(k)$ .

For a point  $z \in (\mathbb{C}^*)^n$ , we denote by  $E_{m,z}$  the closed subspace of the Sobolev space  $H^m(K)$  formed by the restrictions of functions  $v \in H_{\text{loc}}^m(\mathbb{R}^n)$  that satisfy the Floquet condition  $v(x + \gamma) = z^\gamma v(x)$  for any  $\gamma \in \Gamma$ . One can show [19, Theorem 2.2.1] that

$$(2.6) \quad \mathcal{E}_m := \bigcup_{z \in (\mathbb{C}^*)^n} E_{m,z}$$

forms a holomorphic subbundle of the trivial bundle  $(\mathbb{C}^*)^n \times H^m(K)$ . As any infinite dimensional analytic Hilbert bundle over a Stein domain, it is trivializable (see [19, Chapter 1] and Lemma 6 below). One can notice that for  $m = 0$  the bundle  $\mathcal{E}_0$  coincides with the whole  $(\mathbb{C}^*)^n \times L^2(K)$ .

We collect now several statements from Theorem XIII.97 in [35] and Theorems 1.3.2, 1.3.3, 1.5.23 and 2.2.2 in [19]:

**Lemma 6.** (1) *As any infinite dimensional analytic Hilbert bundle over a Stein domain, the bundle  $\mathcal{E}_m$  is analytically trivial.*

(2) *For any nonnegative integer  $m$ , the operator*

$$\mathcal{U} : H^m(\mathbb{R}^n) \rightarrow L^2(T, \mathcal{E}_m)$$

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<sup>2</sup>It is sometimes also called the *Gelfand transform*, due to Gelfand's work [12].

is an isometric isomorphism, where  $L^2(T, \mathcal{E}_m)$  denotes the space of square integrable sections over the complex torus  $T$  of the bundle  $\mathcal{E}_m$ , equipped with the natural topology of a Hilbert space.

(3) Let the space

$$\Theta^m := \left\{ f \in H_{\text{loc}}^m(\mathbb{R}^n) \mid \sup_{\gamma \in \Gamma} \{ \|f\|_{H^m(K+\gamma)} e^{b|\gamma|} \} < \infty, \forall b > 0 \right\}$$

be equipped with the natural Fréchet topology. Then

$$\mathcal{U} : \Theta^m \rightarrow \Gamma((\mathbb{C}^*)^n, \mathcal{E}_m)$$

is a topological isomorphism, where  $\Gamma((\mathbb{C}^*)^n, \mathcal{E}_m)$  is the space of all analytic sections over  $(\mathbb{C}^*)^n$  of the bundle  $\mathcal{E}_m$ , equipped with the topology of uniform convergence on compacta.

(4) Under the transform  $\mathcal{U}$ , the operator

$$P : H^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$$

becomes the operator of multiplication by a holomorphic Fredholm morphism  $P(z)$  between the fiber bundles  $\mathcal{E}_2$  and  $\mathcal{E}_0$ . Here  $P(z)$  acts on the fiber of  $\mathcal{E}_m$  over the point  $z \in T$  as the restriction to this fiber of the operator  $P$  acting between  $H^2(K)$  and  $L^2(K)$ .

Let us now mention another common way of looking at  $P(z)$ . If  $z = \exp ik$ , then commuting with the exponent  $\exp ik \cdot x$  one reduce the bundle  $\mathcal{E}_m$  to the trivial one with the fiber  $H^m(\mathbb{T}^n)$ , where as before  $\mathbb{T}^n = \mathbb{R}^n/\Gamma$ . On the other hand, the operator  $P(z)$  takes the form  $P(x, D + k)$  acting between Sobolev spaces on the torus  $\mathbb{T}^n$ . In other words, the options are either dealing with the restriction of a fixed operator to an analytically “rotating” subspace, or with a polynomial family of operators between fixed spaces.

We will need to see how the structure of the Floquet solutions (see Definition 1), and in general, the structure of functions of Floquet type (2.2) reacts to the Floquet transform. For instance, in the constant coefficient case, where the role of the Floquet solutions is played by the exponential polynomials

$$e^{ik \cdot x} \sum_{|j| \leq N} p_j x^j,$$

such functions are Fourier transformed into distributions supported at the point  $(-k)$ . The next statement shows that under the Floquet transform, each Floquet type function of the form (2.2) corresponds, in a similar way, to a (vector valued) distribution supported at the quasimomentum  $(-k)$ .

Every Floquet type function  $u$  (see (2.2)), being of exponential growth, determines a (continuous linear) functional on the space  $\Theta^0$ . If it satisfies the equation  $Pu = 0$  for a periodic elliptic operator of order  $m$ , then as such a functional it is orthogonal to the range of the dual operator  $P^* : \Theta^m \rightarrow \Theta^0$ . According to Lemma 6, after the Floquet transform any such functional becomes a functional on  $\Gamma((\mathbb{C}^*)^n, \mathcal{E}_0)$ , which is orthogonal to the range of the Fredholm morphism  $P^*(z) : \mathcal{E}_m \rightarrow \mathcal{E}_0$  generated by the dual operator  $P^* : \Theta^m \rightarrow \Theta^0$ . The following auxiliary result describes all such functionals.

**Lemma 7** ([21, Lemma 8]). *A continuous linear functional  $u$  on  $\Theta^0$  is generated by a function of the Floquet form (2.2) with a quasimomentum  $k$  if and only if after the Floquet transform it corresponds to a functional on  $\Gamma((\mathbb{C}^*)^n, \mathcal{E}_0)$  which is a distribution  $\phi$  that is supported at the point  $\nu = \exp(-ik)$ , i.e. has the form*

$$\langle \phi, f \rangle = \sum_{|j| \leq N} \left\langle q_j, \frac{\partial^{|j|} f}{\partial z^j} \Big|_{\nu} \right\rangle \quad f \in \Gamma((\mathbb{C}^*)^n, \mathcal{E}_0),$$

where  $q_j \in L^2(K)$ . The orders  $N$  of the Floquet function (2.2) and of the corresponding distribution  $\phi$  are the same.

Everything discussed so far applies to essentially any elliptic periodic scalar or matrix operators of any order, not necessarily to the ones of the form (2.1) (see [19, 21, 22]). However, there is a special construction that applies only to operators (2.1) and which will play a crucial role in our considerations. Its properties were studied in detail in [1, 26, 34].

Consider the function  $\Lambda(\xi) : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by the condition that the equation

$$Pu = \Lambda(\xi)u$$

has a *positive* Bloch solution of the form

$$(2.7) \quad u_{\xi}(x) = e^{\xi \cdot x} p_{\xi}(x),$$

where  $p_{\xi}(x)$  is  $\Gamma$ -periodic.

**Lemma 8** ([Lemma 12][21]). (1) *The value  $\Lambda(\xi)$  is uniquely determined for any  $\xi \in \mathbb{R}^n$ .*

(2) *The function  $\Lambda(\xi)$  is bounded from above, strictly concave, analytic, and has a nonzero gradient at all points except at its maximum point.*

(3) *Consider the operator*

$$P(\xi) = e^{-\xi \cdot x} P e^{\xi \cdot x} = P(x, D - i\xi)$$



on the torus  $\mathbb{T}^n$ . Then  $\Lambda(\xi)$  is the principal eigenvalue of  $P(\xi)$  with a positive eigenfunction  $p_\xi$ . Moreover,  $\Lambda(\xi)$  is algebraically simple.

- (4) The Hessian of  $\Lambda(\xi)$  is nondegenerate at all points.

One should note that since the function  $\Lambda(\xi)$  is analytic, it is actually defined in a neighborhood of  $\mathbb{R}^n$  in  $\mathbb{C}^n$ . This remark will be used in what follows.

Let us denote

$$(2.8) \quad \Lambda_0 := \max_{\xi \in \mathbb{R}^n} \Lambda(\xi).$$

It follows from [1, 26] that an alternative definition of  $\Lambda_0$  can be

$$(2.9) \quad \Lambda_0 = \sup\{\lambda \in \mathbb{R} \mid \exists u > 0 \text{ such that } (P - \lambda)u = 0 \text{ in } \mathbb{R}^n\},$$

and that in the self-adjoint case  $\Lambda_0$  coincides with the bottom of the spectrum of the operator  $P$ . The common name for  $\Lambda_0$  is *the generalized principal eigenvalue* of the operator  $P$  in  $\mathbb{R}^n$ .

In our main result, we will need to assume that  $\Lambda_0$  is strictly positive. In the self-adjoint case such an assumption has a clear spectral interpretation: the bottom of the spectrum is strictly positive. In the next lemma, we provide some known conditions for the nonnegativity or positivity of  $\Lambda_0$  for not necessarily self-adjoint operators of the form (2.1).

**Lemma 9** ([21, Lemma 13]). *Consider an operator  $P$  of the form (2.1)*

- (1)  $\Lambda_0 \geq 0$  if and only if the operator  $P$  admits a positive (super)solution. This condition is satisfied in particular when  $c(x) \geq 0$ .
- (2)  $\Lambda_0 \geq 0$  if and only if the operator  $P$  admits a positive solution of the form (2.7).
- (3)  $\Lambda_0 = 0$  if and only if the equation  $Pu = 0$  admits exactly one normalized positive solution in  $\mathbb{R}^n$ .
- (4) If  $c(x) = 0$ , then  $\Lambda_0 = 0$  if and only if  $\int_{\mathbb{T}^n} b(x)\psi(x) dx = 0$ , where  $\psi$  is the principal eigenfunction of  $P^*$  on  $\mathbb{T}^n$  (with principal eigenvalue zero). In particular, divergence form operators satisfy this condition.
- (5) Let  $\xi \in \mathbb{R}^n$ , and assume that  $u_\xi(x) = e^{\xi \cdot x} p_\xi(x)$  and  $u_{-\xi}^*$  are positive Bloch solutions of the equations  $Pu = 0$  and  $P^*u = 0$ , respectively. Denote by  $\psi$  the periodic function  $u_\xi u_{-\xi}^*$ . Consider

the function

$$\tilde{b}_i(x) := b_i(x) - 2 \sum_{j=1}^n a_{ij}(x) \{ \xi_j + [p_\xi(x)]^{-1} \partial_j p_\xi(x) \},$$

and denote

$$\gamma = (\gamma_1, \dots, \gamma_n) := \left( \int_{\mathbb{T}^n} \tilde{b}_1(x) \psi(x) \, dx, \dots, \int_{\mathbb{T}^n} \tilde{b}_n(x) \psi(x) \, dx \right).$$

Then  $\Lambda_0 = 0$  if and only if  $\gamma = 0$ .

Let us discuss also some additional properties that will play an important role in the sequel. Assume that  $\Lambda_0 > 0$ . Then Lemma 8 implies that the zero level set

$$(2.10) \quad \Xi := \{ \xi \in \mathbb{R}^n \mid \Lambda(\xi) = 0 \}$$

is a strictly convex compact analytic surface in  $\mathbb{R}^n$  of dimension  $n - 1$ . The manifold  $\Xi$  consists of all  $\xi \in \mathbb{R}^n$  such that the equation  $Pu = 0$  admits a positive Bloch solution  $u_\xi(x) = e^{\xi \cdot x} p_\xi(x)$ . Moreover, the set of all such positive Bloch solutions is the set of all *minimal positive solutions* of the equation  $Pu = 0$  in  $\mathbb{R}^n$  [1, 26]<sup>3</sup>. We denote by  $G$  the convex hull of  $\Xi$ , and by  $\overset{\circ}{G}$  its interior ( $\overset{\circ}{G}$  is nonempty if and only if  $\Lambda_0 > 0$ ).

**Lemma 10** ([21, Lemma 14]). *Suppose that  $\Lambda_0 > 0$ . There exists a neighborhood  $W$  of  $G$  in  $\mathbb{C}^n$  and an analytic function*

$$W \ni \xi \mapsto p_\xi(\cdot) \in H^2(\mathbb{T}^n)$$

*such that for any  $\xi \in W$  the function of  $x$*

$$u_\xi(x) = \exp(\xi \cdot x) p_\xi(x)$$

*is a nonzero Bloch solution of the equation  $Pu = \Lambda(\xi)u$  with a quasi-momentum  $-\mathrm{i}\xi$ . Moreover, one can choose the function  $p$  in such a way that it is positive for all  $\xi \in \Xi$ .*

Comparing  $\Xi$  with the Fermi surface  $F_P$ , one sees that

$$-\mathrm{i}\Xi \subset F_P.$$

The next result specifies further the relation between these two varieties:

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<sup>3</sup>It is also established that a function  $u$  is a positive solution of the equation  $Pu = 0$  in  $\mathbb{R}^n$  if and only if there exists a positive finite measure  $\mu$  on  $\Xi$  such that

$$u(x) = \int_{\Xi} u_\xi(x) \, d\mu(\xi).$$

**Lemma 11** ([21, Lemma 15]). *Let  $\Lambda_0 \geq 0$ . Then*

- (1) *The intersection of the complex Fermi surface  $F_P$  with the tube*

$$(2.11) \quad \mathcal{T} := \{k \in \mathbb{C}^n \mid \operatorname{Im} k = (\operatorname{Im} k_1, \dots, \operatorname{Im} k_n) \in -G\}$$

*coincides with the union of the surface  $-i\Xi$  with its translations by the vectors of the reciprocal lattice  $\Gamma^*$ , i.e. consists of vectors  $k = -i\xi + \gamma$  where  $\xi \in \Xi$  and  $\gamma \in \Gamma^*$ . Moreover, up to a multiplicative constant, any nonzero Bloch solution with a quasimomentum in the above intersection is a positive Bloch solution.*

- (2) *If  $\Lambda_0 > 0$ , then the intersection of  $F_P$  with a sufficiently small neighborhood of  $-i\Xi$  is a (smooth) analytic manifold that coincides with the set of zeros of the function  $\Lambda(ik)$ .*

Analogously to the Floquet surface  $\Phi = \Phi_P$ , we define the surface

$$(2.12) \quad \Psi := \rho(-i\Xi) = \{z \mid z = (\exp \xi_1, \dots, \exp \xi_n), \xi \in \Xi\},$$

and the tubular domain

$$(2.13) \quad V := \rho(\mathcal{T}),$$

where  $\mathcal{T}$  was defined in (2.11). The results of lemmas 10 and 11 can be restated in terms of these new objects:

**Lemma 12.** *Let  $\Lambda_0 \geq 0$ . Then*

- (1)  $\Phi \cap V = \Psi$ .

*If  $\Lambda_0 > 0$ , then*

- (2) *The intersection of  $\Phi$  with a sufficiently small neighborhood of  $\Psi$  is a (smooth) connected analytic manifold.*  
 (3) *The intersections of  $\Phi$  with neighborhoods of the tube  $V$  form a basis of neighborhoods of  $\Psi$  in  $\Phi$ .*  
 (4) *For a sufficiently small neighborhood  $\Phi_\varepsilon$  of  $\Psi$  in  $\Phi$  there exists an analytic function  $p : \Phi_\varepsilon \rightarrow H^2(\mathbb{T}^n)$  such that for any  $z \in \Phi_\varepsilon$  the function of  $x$*

$$u_z(x) = z^x p(z, x)$$

*is a nonzero Bloch solution of the equation  $Pu = 0$ .*

### 3. REPRESENTATION OF SOLUTIONS BY HYPERFUNCTIONS AND DISTRIBUTIONS

The main result of this paper (Theorem 13 below) is analogous to the results of [2, 3] that characterize the classes of solutions of the Helmholtz equation that can be represented by means of distributions

or hyperfunctions on  $S$  (see also the introduction to our paper). In order to state it, we need to introduce a new object. Let us denote by  $h(\omega)$ ,  $\omega \in S^{n-1}$  the indicator function of the convex domain  $G$  introduced in the previous section. Namely,

$$(3.1) \quad h(\omega) := \sup_{\xi \in G} (\omega \cdot \xi),$$

where  $\omega \cdot \xi = \sum_{j=1}^n \omega_j \xi_j$  is the inner product in  $\mathbb{R}^n$ . The next main theorem will be stated in terms of this function.

**Theorem 13.** *Suppose that  $\Lambda_0 > 0$ .*

- (1) *Let  $u$  be a solution of the equation  $Pu = 0$  in  $\mathbb{R}^n$  satisfying for some  $N$  the estimate*

$$(3.2) \quad |u(x)| \leq C(1 + |x|)^N e^{h(x/|x|)|x|}.$$

*Then  $u$  can be represented as*

$$(3.3) \quad u(x) = \langle \mu(\xi), u_\xi(x) \rangle,$$

*where  $u_\xi$  is the analytic positive Bloch solution corresponding to  $\xi \in \Xi$  (see Lemma 10), and  $\mu(\xi)$  is a distribution on  $\Xi$ . The converse statement is also true: for any distribution  $\mu$  on  $\Xi$ , the function  $u(x)$  in (3.3) is a solution of the equation  $Pu = 0$  in  $\mathbb{R}^n$  which satisfies for some  $N$  the growth condition (3.2).*

- (2) *Let  $u$  be a solution of the equation  $Pu = 0$  in  $\mathbb{R}^n$  satisfying for any  $\varepsilon > 0$  the estimate*

$$(3.4) \quad |u(x)| \leq C_\varepsilon \exp [(h(x/|x|) + \varepsilon)|x|],$$

*where  $C_\varepsilon$  is a constant depending only on  $\varepsilon$  and  $u$ . Then  $u$  can be represented as in (3.3) with  $\mu(\xi)$  being a hyperfunction (analytic functional) on  $\Xi$ . The converse statement is also true: for any hyperfunction  $\mu$  on  $\Xi$ , the function  $u(x)$  in (3.3) is a solution of the equation  $Pu = 0$  in  $\mathbb{R}^n$  which satisfies the growth condition (3.4)*

*Proof.* The second statement of the theorem is proven in our paper [21] with Y. Pinchover. So, we concentrate now on the proof of the first one. The proof consists of three major parts: defining appropriate function spaces and interpreting the corresponding class of solutions as functionals; proving Paley-Wiener type theorems for this class of spaces (Lemma 15 below); constructing a specific exact sequence of topological spaces. The last step, i.e. constructing and proving exactness of a sequence (Lemma 17) is usually the most technical one.

Let us make first of all the following remark:

**Remark 14.** Using a standard elliptic argument (Schauder type estimate) and periodicity of the equation, it is standard to show that a solution satisfies for some  $N$  the pointwise growth condition (3.2) if and only if it satisfies for some (different)  $N$  the following  $L_2$  growth condition:

$$(3.5) \quad u(x)(1 + |x|)^{-N} e^{-h(x/|x|)|x|} \in L^2(\mathbb{R}^n).$$

Let us now return to the proof of the theorem. Assume first that a function  $u$  has the representation (3.3) with a distribution  $\mu$ . Then it is obvious that it is a solution of the equation  $Pu = 0$ . We only need to establish the estimate (3.2). Due to compactness of  $\Xi$ , the distribution  $\mu$  can be represented as a finite sum of terms of the form  $D^k \mu_k(\xi)$ , where  $D^k$  is a constant coefficient homogeneous linear differential operator of order  $k$  with respect to  $\xi$  and  $\mu_k$  is a measure on  $\Xi$ . So, it is sufficient to establish (3.2) for such a term only. In other words, we need an estimate of the function  $v(x) = \langle \mu_k(\xi), D_\xi^k u_\xi(x) \rangle$ . According to Lemma 7,  $D_\xi^k u_\xi(x)$  is an analytically depending on  $\xi \in \Xi$  Floquet solution of  $Pu = 0$  of order  $k$ . This means that it satisfies an estimate of the type (3.2) with  $N = k$ . Then the estimate for  $v(x)$  follows, since  $\mu_k$  is a finite measure. Hence  $u(x)$ , being the sum of a finitely many such terms, also satisfies (3.2).

Suppose now that  $u$  satisfies (3.2). We need to prove that  $u$  can be represented as in (3.3). In order to do so, we need first to interpret this class of solutions in dual terms.

Consider the following Fréchet spaces of test functions:

$$W_m := \{ \phi \in H_{\text{loc}}^m(\mathbb{R}^n) \mid \langle \phi \rangle_{m,N} < \infty \quad \forall N > 0 \},$$

where

$$\langle \phi \rangle_{m,N} := \sup_{\gamma \in \Gamma} \left\{ \|\phi\|_{H^m(K+\gamma)} (1 + |\gamma|)^N e^{h(\gamma/|\gamma|)|\gamma|} \right\}.$$

The operator  $P^*$  clearly maps continuously  $W_2$  into  $W_0$ . It is also clear that due to (3.2), the linear functional

$$\langle u, \phi \rangle := \int_{\mathbb{R}^n} u(x) \phi(x) dx$$

is continuous on the space  $W_0$ . Since  $Pu = 0$ , Schauder elliptic estimates together with the periodicity of the operator show that estimates similar to (3.2) hold also for the derivatives of  $u$ . One observes by a simple argument that  $u$  is a continuous functional on  $W_0$ , which annihilates the range of the dual operator  $P^* : W_2 \rightarrow W_0$ . Now we can apply Floquet theory arguments analogous to the ones used in [19, Section 3.2] or in [21] to obtain (3.3). However, some technical details needed

in the cases considered in [19, 21] and in this paper are significantly different, so we provide the details of this derivation.

First of all, we need to obtain a Paley-Wiener type theorem for the Floquet transform in the spaces  $W_m$ . Let us denote by  $V^*$  the tube that consists of all points  $z \in (\mathbb{C}^*)^n$  such that  $z^{-1} = (z_1^{-1}, \dots, z_n^{-1}) \in V$ , where the tube  $V$  is defined in (2.13). We introduce the space  $A^\infty(V^*)$  of holomorphic functions on the tube  $V^*$  that are infinitely differentiable up to its boundary  $\partial V^*$ . Analogously, if  $\mathcal{E}$  is a holomorphic Banach bundle in a neighborhood of  $V^*$ , we denote by  $A^\infty(V^*, \mathcal{E})$  the space of sections of  $\mathcal{E}$  over the (closed) tube  $V^*$  that are holomorphic in the interior and infinitely differentiable up to the boundary of  $V^*$ . This space is equipped with the natural Fréchet space topology. The following statement is a Paley-Wiener type theorem for the transform  $\mathcal{U}$  in the spaces  $W_m$ .

**Lemma 15.** (1) *The operator*

$$\mathcal{U} : W_m \rightarrow A^\infty(V^*, \mathcal{E}_m).$$

*is a topological isomorphism.*

(2) *Under the transform  $\mathcal{U}$ , the operator*

$$P^* : W_2 \rightarrow W_0$$

*becomes the operator  $\mathcal{P}(z)$  of multiplication by a holomorphic Fredholm morphism between the fiber bundles  $\mathcal{E}_2$  and  $\mathcal{E}_0$ :*

$$A^\infty(V^*, \mathcal{E}_2) \xrightarrow{\mathcal{P}(z)} A^\infty(V^*, \mathcal{E}_0).$$

*Here  $\mathcal{P}(z)$  acts on each fiber of  $\mathcal{E}_2$  as the restriction to this fiber of the operator  $P^*$  acting between  $H^2(K)$  and  $L^2(K)$ .*

Before proving this lemma, we first obtain the following auxiliary statement:

**Lemma 16.** *Let  $H$  be a complex Hilbert space and  $W(H)$  be the Fréchet space of sequences  $f = \{f_\gamma\}$ ,  $f_\gamma \in H$ ,  $\gamma \in \Gamma$  such that the semi-norm*

$$\phi_N(f) := \sup_{\gamma \in \Gamma} \{ \|f_\gamma\|_H (1 + |\gamma|)^N e^{h(\gamma/|\gamma|)|\gamma|} \}$$

*is finite for any  $N$ . Here, as before,  $h$  is the indicator function (3.1).*

*Then a sequence  $f = \{f_\gamma\}$  belongs to  $W(H)$  if and only if the function*

$$(3.6) \quad \widehat{f}(z) := \sum_{\gamma \in \Gamma} f_{-\gamma} z^\gamma$$

*belongs to  $A^\infty(V^*, H)$ . The mapping  $f \mapsto \widehat{f}$  is an isomorphism of the space  $W(H)$  onto  $A^\infty(V^*, H)$ .*

**Proof:** Let  $f \in W(H)$ . We will show that the series (3.6) converges uniformly on  $V^*$  as a series of  $H$ -valued functions on  $V^*$ . This will imply that  $\hat{f}$  is analytic in  $V^*$  and continuous up to the boundary. Then we will check that the same holds for the derivatives of the series, which will imply that  $\hat{f} \in A^\infty(V^*, H)$ .

Taking into account that any  $z \in V^*$  can be represented as  $z = e^{-ik}$  with  $\text{Im}k \in G$ , and thus  $\text{Im}k \cdot \gamma \leq h(\gamma/|\gamma|)|\gamma|$ , we can estimate

$$\begin{aligned}
 \|\hat{f}(z)\| &\leq \sum_{\gamma \in \Gamma} \|f_{-\gamma}\| e^{-\text{Im}k \cdot \gamma} = \sum_{\gamma \in \Gamma} \|f_\gamma\| e^{\text{Im}k \cdot \gamma} \\
 (3.7) \quad &\leq \sum_{\gamma \in \Gamma} (1 + |\gamma|)^{-n-1} \|f_\gamma\| (1 + |\gamma|)^{n+1} e^{h(\gamma/|\gamma|)|\gamma|} \\
 &\leq \left[ \sum_{\gamma \in \Gamma} (1 + |\gamma|)^{-n-1} \right] \phi_{n+1}(f).
 \end{aligned}$$

Since the series  $\sum_{\gamma \in \Gamma} (1 + |\gamma|)^{-n-1}$  converges, this implies the analyticity in  $V^*$  and continuity up to the boundary of  $\hat{f}(z)$ . Multiple differentiation with respect to  $k$  amounts to multiplying the coefficients of (3.6) by a polynomial with respect to  $\gamma$  factor. Due to the definition of the space  $W(H)$ , one can get an estimate from above similar to (3.7), but with the seminorm  $\phi_{n+d+1}(f)$  instead of  $\phi_{n+1}(f)$ , where  $d$  is the order of differentiation. Thus, in fact the function is infinitely smooth up to the boundary. These estimates also prove that the mapping  $f \in W(H) \mapsto \hat{f} \in A^\infty(V^*, H)$  is continuous.

Let us now prove the surjectivity of this mapping. Assume that  $s(z) \in A^\infty(V^*, H)$ . Let  $z = \exp ik$ , then  $s$  as a function of  $k$  is periodic with respect to the reciprocal lattice  $\Gamma^*$ . Expanding it into the Fourier series, we get

$$(3.8) \quad s(z) = \sum_{\gamma \in \Gamma} s_{-\gamma} z^\gamma = \sum_{\gamma \in \Gamma} s_\gamma z^{-\gamma},$$

where  $s_{-\gamma} \in H$ . We need to show now that  $\{s_\gamma\} \in W(H)$ . For this purpose, we use the standard formulas for the Fourier coefficients:

$$s_\gamma = \frac{1}{(2\pi)^n} \int_B s(e^{i(\beta - i\alpha)}) e^{i(\beta - i\alpha) \cdot \gamma} d\beta, \quad \forall \alpha \in G,$$

where  $B$  is the first Brillouin zone, and we write  $z = \exp ik = \exp[i(\beta - i\alpha)]$ ,  $\alpha \in G$ .

Integrating by parts  $l$  times with respect to  $\beta$ , where  $l = (l_1, \dots, l_n)$  is a multi-index, we obtain analogously

$$(3.9) \quad s_\gamma = \frac{(-i\gamma)^{-l}}{(2\pi)^n} \int_B \frac{\partial^l s}{\partial \beta^l} (e^{i(\beta-i\alpha)}) e^{i(\beta-i\alpha)\cdot\gamma} d\beta \quad \forall \alpha \in G.$$

Now straightforward norm estimate in (3.9) gives

$$(3.10) \quad \|s_\gamma\|_H \leq C \max_{z \in V^*} \left\| \frac{\partial^l s}{\partial \beta^l}(z) \right\|_H \gamma^{-l} e^{-\alpha \cdot \gamma}$$

for any multi-index  $l$  and any  $\alpha \in G$ . Optimizing with respect to  $\alpha \in G$ , we get

$$(3.11) \quad \|s_\gamma\|_H \leq C_N (1 + |\gamma|)^{-N} e^{-h(\gamma/|\gamma|)|\gamma|}$$

for any  $N$ . This means that  $f := \{s_\gamma\}$  belongs to  $W(H)$  and by its construction  $\hat{f} = s(z)$ . This proves Lemma 16.  $\square$

Let us now complete the proof of Lemma 15.

We start proving the first claim of the lemma. Let a function  $F(x)$  belong to  $W_m$ . Consider a sequence  $f = \{f_\gamma\}$  of elements of  $H^m(K)$  defined as follows:

$$f_\gamma(x) = F(x + \gamma) \quad x \in K, \gamma \in \Gamma.$$

Then clearly the condition  $F \in W_m$  is equivalent to two conditions: the first one that  $f \in W(H^m(K))$ , and second that  $F \in H_{\text{loc}}^m(\mathbb{R}^n)$ , i.e. that the functions  $f_\gamma$  defined on shifted copies of the fundamental domain  $K$ , fit smoothly across the boundaries.

Analogously, the requirement that a section  $\phi$  belongs to  $A^\infty(V^*, \mathcal{E}_m)$  consists of two conditions. The first one that  $\phi \in A^\infty(V^*, H^m(K))$  and the second that it is a section of the subbundle  $\mathcal{E}_m \subset V^* \times H^m(K)$ .

We can notice now that the Floquet transform on  $W_m$  is the restriction of the transform  $f \mapsto \hat{f}$  of Lemma 16 from the larger space  $W(H^m(K))$ . Thus, Lemma 16 claims that this transform is an isomorphism of  $W(H^m(K))$  onto  $A^\infty(V^*, H^m(K))$ . On the other hand, the second conditions: the fitting of  $f_\gamma$  across the boundaries and being a section of the subbundle  $\mathcal{E}$ , are intertwined by the Floquet transform, according to the first statement of Lemma 6. This proves the first claim of the lemma.

Now, the second claim of Lemma 15 follows from the third one of Lemma 6. Lemma 15 is proven.  $\square$

Let us now return to the proof of Theorem 13. We remind the reader that we have a solution  $u$  with the estimate (3.2), for which we need to prove the representation (3.3). Let us apply the Floquet transform  $\mathcal{U}$ . Then the image  $\mathcal{U}u$  of the solution  $u$  under the Floquet transform is a



continuous linear functional on  $A^\infty(V^*, \mathcal{E}_0)$ , which is in the cokernel of the operator

$$A^\infty(V^*, \mathcal{E}_2) \xrightarrow{\mathcal{P}(z)} A^\infty(V^*, \mathcal{E}_0).$$

This, indeed is a one-to-one correspondence between solutions of the required class and such functionals. Thus, we need to describe all such functionals. Let  $u_z(\cdot) = z^x p(z, \cdot)$  be the Bloch solution of the equation  $Pu = 0$  introduced in Lemma 12. We will also employ the space  $C^\infty(\Psi)$  with the standard topology, where the smooth variety  $\Psi$  is introduced in (2.12). Consider the mapping

$$t : A^\infty(V^*, \mathcal{E}_0) \rightarrow C^\infty(\Psi)$$

that for a section  $f(z, x) \in A^\infty(V^*, \mathcal{E}_0)$  of the bundle  $\mathcal{E}_0$  produces

$$t_f(z) = \langle f(z^{-1}, \cdot), u_z(\cdot) \rangle = \int_{\mathbb{T}^n} f(z^{-1}, x) u_z(x) dx.$$

Here  $z^{-1} = (z_1^{-1}, \dots, z_n^{-1})$ .

As we will see soon, the following lemma will finish the proof of the theorem:

**Lemma 17.** *The mapping  $t$  is a topological homomorphism and the following sequence is exact:*

$$(3.12) \quad A^\infty(V^*, \mathcal{E}_2) \xrightarrow{\mathcal{P}(z)} A^\infty(V^*, \mathcal{E}_0) \xrightarrow{t} C^\infty(\Psi) \rightarrow 0.$$

**Proof of the lemma.** Continuity of  $\mathcal{P}(z)$  is already established. Continuity of  $t$  is obvious. The complex property of the sequence (3.12) (i.e. that  $t\mathcal{P}(z) = 0$ ) follows from the construction of  $t$ . Thus, the only thing that requires proof is exactness in the second and third terms. The topological homomorphism property will follow then from exactness and the open mapping theorem. So, we only need to prove that: i) any section  $\phi \in A^\infty(V^*, \mathcal{E}_0)$  such that  $t\phi = 0$  belongs to the range of  $\mathcal{P}(z)$  and ii) any function  $f \in C^\infty(\Psi)$  is in the range of  $t$ .

Let us start with the first of these tasks. So, let  $\phi \in A^\infty(V^*, \mathcal{E}_0)$  be such that  $t\phi = 0$ . Consider the inverse  $\mathcal{P}^{-1}(z)$  to the morphism  $\mathcal{P}(z)$ . It is defined (and hence holomorphic) in a neighborhood  $V_\epsilon^*$  of the tube  $V^*$ , except for an analytic submanifold, whose intersection with  $V^*$  is  $\Psi$  (see Lemma 12). Let us consider the function  $f = \mathcal{P}^{-1}(z)\phi(z)$ . The only thing now to prove is that this function does not have any singularities along  $\Psi$ . This is a local question, so let us return in a neighborhood of a point of  $\Psi$  to the quasi-momenta coordinates  $k$  and consider the structure of the inverse  $\mathcal{P}^{-1}(z)$ . As it was shown in the proof of [21, Lemma 21], the inverse has the form  $B(k)/\Lambda(k)$ ,

where  $B(k)$  is an analytic operator-valued function. This means that  $f(k) = (B(k)\phi(k))/\Lambda(k)$ . The condition  $t\phi = 0$  guarantees that the numerator  $g(k) = B(k)\phi(k) \in A^\infty(V^*, H)$  vanishes on  $\Psi$ , where  $H$  is a Hilbert space. Our goal is to prove that this is sufficient for its smooth divisibility (on  $\partial V^*$ ) by  $\Lambda$ . We recall here that  $\Lambda$  is analytic in a vicinity of  $\partial V^*$  and has simple zeros along  $\Psi$  (Lemmas 8 and 11). We notice that it is sufficient to prove this for scalar functions, i.e. for  $H = \mathbb{C}$ . This can be justified in many different ways. For instance, the statement is local, and locally, due to the Fredholm nature of the morphism  $\mathcal{P}(z)$ , one can project the problem onto a finite dimensional subspace, using a lemma by M. Atiyah [5] (see also [36, Lemma 2.1] and [19, Lemma 1.2.11 and Theorem 1.3.9]), which will reduce it to a finite dimensional, and thus also to scalar case. So, we will assume in this part of the proof that  $g \in A^\infty$  is a scalar function. According to a result of [17, 27] (see also [29, 31] and [29, Theorem 1.1' in Ch. VI]), it is sufficient to check the divisibility at each point of  $\Psi$  on the level of formal Taylor series. So, let us pick a point  $k$  of  $\Psi$  and introduce coordinates  $x \in \mathbb{R}^{n-1}$  in the tangent space  $T_k(\Psi) \in i\mathbb{R}^n$ . The complexification  $T_k^c(\Psi)$  of this tangent space is a part of the tangent space to the boundary of the tube. Let us chose coordinates  $y \in \mathbb{R}^{n-1}$  in  $T_k^c(\Psi) \cap \mathbb{R}^n$  that correspond to the coordinates  $x$  in  $T_k(\Psi)$ . An extra coordinate  $t$  in  $T_k(\Psi) \cap \mathbb{R}^n$  is required to obtain the whole tangent space  $T_k(\partial V^*)$ . Let us denote by  $\widehat{g}(x, y, t)$  and  $\widehat{\Lambda}(x, y, t)$  the formal Taylor series of  $g$  and  $\Lambda$  at the point  $k$ . Then we know that  $\widehat{g}(x, 0, 0) = 0$  and  $\widehat{\Lambda}(x, 0, 0) = 0$  (formal power series versions of vanishing of functions  $g$  and  $\Lambda$  on  $\Psi$ ). Recall that  $\widehat{g}(x, y, t)$  is the series for a CR-function  $g$  on the boundary (since  $g$  is the boundary value of an analytic function). This means that  $\widehat{g}(x, y, t)$  satisfies Cauchy-Riemann conditions with respect to the variable  $z = x + iy \in \mathbb{C}^{n-1}$ . Then uniqueness of analytic continuation<sup>4</sup> claims that  $\widehat{g}(x, 0, 0) = 0$  for all  $x$  implies  $\widehat{g}(x, y, 0) = 0$  for all  $(x, y)$ . The same is true for  $\widehat{\Lambda}$ , due to analyticity of  $\Lambda$ . Now, in coordinates  $z = x + iy, t$  we are dealing with the formal series  $\widehat{g}(z, t)$  and  $\widehat{\Lambda}(z, t)$ , both of which vanish at  $t = 0$  and such that  $\widehat{\Lambda}$  has zero of first order at  $t = 0$ . Then, vanishing of  $\widehat{g}(z, 0)$  guarantees divisibility in formal series of  $\widehat{g}$  by  $\widehat{\Lambda}$ . As it was explained above, this implies smooth divisibility of  $g$  by  $\Lambda$  and thus finishes the proof of exactness in the second term of the sequence (3.12).

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<sup>4</sup>The uniqueness of analytic continuation in this power series setting is straightforward to derive algebraically directly from the Cauchy-Riemann conditions for power series.

Let us now prove the exactness in the third term of the sequence. First of all, we notice that the vector-function  $u_z$ , defined on  $\Psi$  only, can be extended to an analytic vector-function (which we will denote the same way) on  $V_\epsilon^*$  for some small epsilon. Indeed, as it is shown in [21],  $V_\epsilon^*$  is a Stein manifold. Then, according to the Corollary 1 from the Bishop's theorem 3.3 in [36] (see the original theorem in [7]), the restriction mapping to an analytic subset of a Stein variety is surjective. Thus, the required extension of  $u_z$  exists. Let also  $v(z)$  be a holomorphic family such that  $tv(z)|_\Psi = 1$  (it is not hard to prove the existence of such a family). Consider a function  $\phi(z) \in C^\infty(\Psi)$ . Notice that the domain  $V^*$  is strictly pseudo-convex and the complexifications of the tangent spaces to the submanifold  $\Psi \subset \partial V^*$  are parts of the tangent spaces to  $\partial V^*$ . Thus,  $\Psi$  and  $\partial V^*$  satisfy the conditions of [9] needed for  $\Psi$  to be an  $A^\infty$  interpolation variety, and hence the restriction mapping  $A^\infty(V^*) \mapsto C^\infty(\Psi)$  is surjective. Hence, there exists a function  $\psi \in A^\infty(V^*)$  such that  $\psi|_\Psi = \phi$ . Now taking  $f = \psi(z)v(z) \in A^\infty(V^*, \mathcal{E}_0)$  guarantees that  $tf = \phi$ . This finishes the proof of the lemma.  $\square$

It is easy now to finish the proof of the theorem. Indeed, after the Floquet transform solution  $u$  becomes a continuous linear functional on  $A^\infty(V^*, \mathcal{E}_0)$  that annihilates the range of the operator of multiplication by  $\mathcal{P}(z)$ . Lemma 17 implies that such a functional can be pushed down to the space  $C^\infty(\Psi)$ . Any such functional is a distribution  $\mu$ . Hence, the action  $\langle u, \phi \rangle$  of the functional  $u$  on a function  $\phi \in W_0$  can be obtained as

$$\langle u, \phi \rangle = \langle \mu(z), t(z)(\mathcal{U}\phi) \rangle.$$

Applying now the explicit formulas for the transforms  $\mathcal{U}$  and  $t$ , one arrives to the representation (3.3). Indeed,

$$\begin{aligned} (3.13) \quad t_{(\mathcal{U}\phi)}(z) &= \int_K \mathcal{U}\phi(z^{-1}, x) u_z(x) dx \\ &= \sum_{\gamma \in \Gamma} \int_{K-\gamma} \phi(x) z^{-\gamma} u_z(x + \gamma) dx \\ &= \int_{\mathbb{R}^n} \phi(x) u_z(x) dx. \end{aligned}$$

In this calculation we used the property of the Bloch solutions

$$u_z(x + \gamma) = z^\gamma u_z(x).$$

Therefore,

$$\langle u, \phi \rangle = \langle \langle \mu(z), u_z \rangle, \phi \rangle,$$

which concludes the proof of the theorem.  $\square$

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, USA

*E-mail address:* kuchment@math.tamu.edu